

# Trivial $I^\tau$ fibrations of the multiplication maps for monads $\mathbb{O}$ and $\mathbb{OH}$

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## Abstract

In this paper we further investigate the geometry of monads  $\mathbb{O}$  of order-preserving functionals and  $\mathbb{OH}$  of positively homogeneous functionals. We prove that for any  $X \in \text{Comp}$  with  $w(X) = \tau$  the map  $\mu_F X$ , where  $F \in \{O, OH\}$ , is homeomorphic to trivial  $I^\tau$ -fibration if and only if  $X$  is openly generated  $\chi$ -homogeneous compactum.

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**Introduction.** The geometric properties of different functors have been studied extensively over the past few decades [5]. The researches concern studying the question of how functors affect properties of spaces and maps between them as well as the investigation of properties of maps involved in the structures generated by functors (i.e. monad multiplication maps, structural mappings of algebras).

In the present paper we continue the investigation of geometric properties of monads which have functional representation, that is, which can be embedded in monad  $\mathbb{V}$ . This research concerns the monads  $\mathbb{O}$  and  $\mathbb{OH}$  generated by functors of order-preserving and positively homogeneous functionals respectively and is to answer the question when multiplication maps for these monads are trivial fibrations with fibers homeomorphic to the Tychonov cube.

The results of M. Zarichnyi on the inclusion hyperspaces monad (see [14]), for instance, show that for a continuum  $X$  the multiplication map  $\mu_G X$  for this monad is homeomorphic to the projection map  $pr_{G(X)} : I^\tau \times G(X) \rightarrow G(X)$  iff  $X$  is openly generated and  $\chi$ -homogeneous. In this research we obtain a similar condition for multiplication maps of monads  $\mathbb{O}$  and  $\mathbb{OH}$  ( $X$  is not necessarily connected in our case). It is interesting whether there is some kind of a general result of this type (in case of submonads of  $\mathbb{V}$ , for example).

**Definitions and facts.** In this section we shall recall some necessary definitions and results from infinite-dimensional topology as well as define the objects of our investigation - monads of order-preserving and positively homogeneous functionals and name some of their properties.

Since in what follows we will deal with endofunctors in the category  $\text{Comp}$ , we assume all spaces to be compact Hausdorff (briefly compacta) and mappings to be continuous.

By  $w(X)$  we denote the weight of a space  $X$ , and by  $\chi(x, X)$  the character at a point  $x \in X$ . We call  $X$   $\chi$ -homogeneous if for every  $x, y \in X$  we have  $\chi(x, X) = \chi(y, X)$ .

We say that a space  $X$  is a *retract* of  $Y$ , where  $X \subset Y$ , if there exists a map  $r : Y \rightarrow X$  with  $r|_X = \text{id}_X$ . The space  $X$  is called an *absolute retract* (briefly an *AR*), if for every embedding  $i : X \hookrightarrow Y$  the subspace  $i(X)$  is a retract of  $Y$ .

Recall that a  $\tau$ -system, where  $\tau$  is any cardinal number, is a continuous inverse system consisting of compacta of weight  $\leq \tau$  and epimorphisms over a  $\tau$ -complete indexing set. As usual,  $\omega$  stands for the countable cardinal number. A compactum  $X$  is called *openly generated*, if it can be represented as the limit of some  $\omega$ -system with open bonding mappings [11].

By  $C(X)$ , where  $X \in \text{Comp}$ , we denote the Banach space of all continuous real-valued functions on  $X$  with the sup-norm  $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$ . By  $c_X$ , where  $c \in \mathbb{R}$ , we denote the constant function:  $c_X(x) = c$  for all  $x \in X$ .

Let  $\nu : C(X) \rightarrow \mathbb{R}$  be a functional. We say that  $\nu$  is 1) normed, if  $\nu(1_X) = 1$ ; 2) weakly additive, if for any  $\phi \in C(X)$  and  $c \in \mathbb{R}$  we have  $\nu(\phi + c_X) = \nu(\phi) + c$ ; 3) order-preserving, whenever for any  $\varphi, \psi \in C(X)$  such that  $\varphi(x) \leq \psi(x)$  for all  $x \in X$  (i.e.  $\varphi \leq \psi$ ) the inequality  $\nu(\varphi) \leq \nu(\psi)$  holds; 4) positively homogeneous, if for any  $\varphi \in C(X)$  and any real  $t \geq 0$  we have  $\nu(t\varphi) = t\nu(\varphi)$ .

Now for any  $X \in \text{Comp}$  denote  $V(X) = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$ . For any mapping  $f : X \rightarrow Y$  let  $V(f)$  be mapping such that  $V(f)(\nu)(\varphi) = \nu(\varphi \circ f)$  for any  $\nu \in V(X)$ ,  $\varphi \in C(Y)$ . Defined that way,  $V$  forms a covariant functor in category  $\text{Comp}$ .

For any space  $X$  by  $O(X)$  denote the set of functionals satisfying 1)–3) (order-preserving functionals), by  $OH(X)$  – the set of all functionals on  $C(X)$ , which satisfy properties 1)–4) (positively homogenous functionals). Let  $F$  stand for one of  $O, OH$ . The space  $F(X)$  is considered as the subspace of  $V(X)$ . For any function  $f : X \rightarrow Y$  the map  $F(f) : F(X) \rightarrow F(Y)$  is the restriction of  $V(f)$  on the respective space  $F(X)$ . Then  $F$  forms covariant functor in  $\text{Comp}$ , which is the subfunctor of  $V$ . Note that both  $O$  and  $OH$  are weakly normal functors.

A *monad* in the category  $\text{Comp}$  is a triple  $\mathbb{F} = (F, \eta, \mu)$ , where  $\eta : \text{Id}_{\text{Comp}} \rightarrow F$  and  $\mu : F^2 \rightarrow F$  are natural transformations such that the following equalities hold: 1)  $\mu X \circ \eta F(X) = \mu X \circ F(\eta X) = \text{id}_{F(X)}$ ; 2)  $\mu X \circ \mu F(X) = \mu X \circ F(\mu X)$  [3].

The abovementioned functors generate monads. If  $F$  is one of  $V, O, OH$ , the identity and multiplication maps are defined as follows. The natural transformation  $\eta : \text{Id}_{\text{Comp}} \rightarrow F$  is given by  $\eta X(x)(\varphi) = \varphi(x)$  for any  $x \in X$  and  $\varphi \in C(X)$ , and the natural transformation  $\mu : F^2 \rightarrow F$  given by  $\mu X(\nu)(\varphi) = \nu(\pi_\varphi)$ , where  $\pi_\varphi : F(X) \rightarrow \mathbb{R}$ ,  $\pi_\varphi(\lambda) = \lambda(\varphi)$ .

Results on categorical and topological properties of functors  $O$  and  $OH$  can be found in [2], [6] [7], [8], [9]. In particular, we'll use the following statements from about functors  $O$  and  $OH$ :

**Theorem 1.1.** [6], [7] *Functors  $O$  and  $OH$  are open.*

**Theorem 1.2.** [6], [7], [9] *Let  $F \in \{O, OH\}$ . Then 1)  $\mu_F X$  is soft iff  $X$  is openly generated; 2)  $F(X) \in \text{AR}$  iff  $X$  is openly generated.*

**Theorem 1.3.** [8] *Let  $f : X \rightarrow Y$  be open.  $O(f)$  has a degenerate fiber if and only if  $f$  has.*

**Theorem 1.4.** [8] *An openly generated compactum  $X$  is  $\chi$ -homogeneous if and only if  $O(X)$  is.*

We also note that the analogous to theorems 1.3 and 1.4 statements hold in case of functor  $OH$  and their proofs are just the same as in case of  $O$ .

Finally, let us recall the definition of an  $I^\tau$ -fibration and the criteria of an  $I^\tau$ -fibration.

A map  $f : X \rightarrow Y$  is called an  $I^\tau$ -fibration if it is homeomorphic to the projection map  $p_Y : I^\tau \times Y \rightarrow Y$ . Note that a map with all fibers homeomorphic to  $I^\tau$  is not necessarily an  $I^\tau$ -fibration (see [1], [13] for counterexamples).

The following theorem is the well-known Toruńczyk-West criterion of a  $Q$ -fibration (by  $Q$  we denote the Hilbert cube  $[0, 1]^\omega$ ):

**Theorem 1.5.** ([13]) *A soft mapping  $f : X \rightarrow Y$  of metric  $AR$ -compacta is homeomorphic to  $Q$ -fibration if and only if it satisfies the condition of disjoint approximation: for any  $\varepsilon > 0$  there are mappings  $g_1, g_2 : X \rightarrow X$  such that  $g_1(X) \cap g_2(X) = \emptyset$ ,  $d(g_i, id_X) < \varepsilon$ ,  $f \circ g_i = f$ .*

In case of an arbitrary  $\tau$ , the criterion of an  $I^\tau$ -fibration contains the generalization of the condition of the Toruńczyk-West theorem.

Let us give the necessary definitions first. Through  $cov_\lambda(X)$  denote the family of all coverings of cardinality  $\leq \lambda$  which consist of sets which are the intersections of no more than  $\lambda$  sets which are in turn are the unions of no more than  $\lambda$  co-zero sets in  $X$ .

A mapping  $f : X \rightarrow Y$  satisfies the condition of *disjoint  $\lambda$ -approximation* [1] if for any cover  $\Omega \in cov_\lambda(X)$  there exist mappings  $g_1, g_2 : X \rightarrow X$  with disjoint images,  $\Omega$ -close to  $id_X$  and with  $f \circ g_i = f$ .

**Theorem 1.6.** ([1]). *A soft mapping  $f : X \rightarrow Y$  between  $AR$ -compacta with fibers of weight  $\leq \tau$  is a trivial  $I^\tau$ -fibration if and only if  $f$  satisfies the condition of disjoint  $\lambda$ -approximation for any  $\lambda < \tau$ .*

The following statement provides the sufficient condition under which a map satisfies the condition of disjoint  $\lambda$ -approximation.

**Lemma 1.1** ([10]). *Let  $f : X \rightarrow Y$  be the limit projection  $p_1$  of a  $\lambda$ -spectrum  $\{X_\alpha, p_\alpha, \mathcal{A}\}$  such that the index set  $\mathcal{A}$  has the least element 1, all limit projections allow two disjoint sections. Then  $f$  satisfies the condition of disjoint  $\lambda$ -approximation.*

**2. The main results.** For the sake of convenience, we consider the cases  $\tau = \omega$  and  $\tau > \omega$  separately. Let's consider the case of the countable  $\tau$  first.

Define the metric on the space  $O(X)$  for any metrizable compactum  $X$  the following way. In case  $X$  is metrizable, the space  $C(X)$  of all continuous functions over  $X$  is separable. Choose any dense in  $C(X)$  countable set  $\{\varphi_i\}_{i \in \mathbb{N}}$ . We can assume that the function  $0_X$  is not in  $\{\varphi_i\}_{i \in \mathbb{N}}$ . Put

$d_O(\lambda, \nu) = \sum_{i=1}^{\infty} \frac{|\lambda(\varphi_i) - \nu(\varphi_i)|}{\|\varphi_i\| \cdot 2^i}$ . Then  $d_O$  is an admissible metric on  $O(X)$ . Indeed, take any  $B_\varepsilon(\nu) = \{\lambda \in O(X) | d_O(\lambda, \nu) < \varepsilon\}$ . Choose number  $n_0 \in \mathbb{N}$  such that the inequality  $\sum_{i=n_0}^{\infty} \frac{1}{2^{i-1}} < \frac{\varepsilon}{2}$  holds. Then  $O(\nu; \varphi_1, \dots, \varphi_{n_0}; \frac{\varepsilon}{2} \cdot (\frac{1}{\sum_{i=n_0}^{\infty} \frac{1}{\|\varphi_i\| \cdot 2^i}})) \subset B_\varepsilon$ . Hence,  $d_O$  generates the topology on  $O(X)$ .

Before coming to the proof of the theorem let us recall how to extend an order-preserving functional over a single function (see lemma 2 in [7]).

Suppose that the set  $A \subset C(X)$  is such that  $0_X \in A$ ,  $\varphi + c_X \in A$  for any  $\varphi \in A$  and  $c \in \mathbb{R}$ . Consider any order-preserving functional  $\nu$  on  $A$  and some function  $\psi \in C(X) \setminus A$ . If we want to extend  $\nu$  over the space  $A \cup \{\psi + c_X | c \in \mathbb{R}\}$ , the possible values  $\nu(\psi)$  are in the segment  $[\sup\{\nu(\varphi) | \varphi \in A, \varphi \leq \psi\}, \inf\{\nu(\varphi) | \varphi \in A, \varphi \geq \psi\}]$  and only they.

**Theorem 2.1.** *The mapping  $\mu_O X$  is a  $Q$ -fibration for any metrizable space  $X$  which contains more than one point.*

*Proof.* Assume  $X$  is metrizable and not one-point. To prove our theorem, we'll use the Toruńczyk-West criterion (theorem 1.5). It means that for any  $\varepsilon > 0$  we have to find two mappings  $g_1, g_2 : O^2(X) \rightarrow O^2(X)$  which are both  $\varepsilon$ -close to  $id_{O^2(X)}$  and preserve the fibers of  $\mu_O X$ .

Choose some dense in  $C(O(X))$  countable set  $\{\Phi_i\}_{i \in \mathbb{N}}$  that does not contain constant functions. Fix any  $\varepsilon > 0$ . There exists some  $n_0 \in \mathbb{N}$  such that  $\sum_{i=n_0}^{\infty} \frac{|\Lambda(\Phi_i) - M(\Phi_i)|}{\|\Phi_i\| \cdot 2^i} < \frac{\varepsilon}{2}$  for any  $\Lambda, M \in O^2(X)$ .

Now for any function  $\Phi_i$ ,  $i = \overline{1, n_0}$  pick two points  $s_i, v_i \in O(X)$  such that  $\Phi_i(s_i) = \sup\{\Phi_i(x) | x \in O(X)\}$  and  $\Phi_i(v_i) = \inf\{\Phi_i(x) | x \in O(X)\}$ . Denote  $S_0 = \{s_i | i = \overline{1, n_0}\}$ ,  $I_0 = \{v_i | i = \overline{1, n_0}\}$ . Since all  $\Phi_i$  are continuous, for  $\varepsilon_1 = \frac{\varepsilon}{b}$ , where  $b = 8 \sum_{i=1}^{n_0} \frac{1}{\|\Phi_i\| \cdot 2^i}$ , we can find  $\delta > 0$  such that  $B_\delta(s) \cap B_\delta(s') = \emptyset$  for any distinct  $s, s' \in I_0 \cup S_0$  and for any  $s \in S_0 \cup I_0$  and any  $x \in B_\delta(s)$   $\Phi_i(x) \in (\Phi_i(s) - \varepsilon_1, \Phi_i(s) + \varepsilon_1)$  for all  $i \in \{1, \dots, n_0\}$ .

Put  $F_0 = \{\Phi_1, \dots, \Phi_{n_0}\}$ . Take any  $s \in I_0 \cap S_0$ . Let  $\Phi_{i_1}, \dots, \Phi_{i_k}$  and  $\Phi_{j_1}, \dots, \Phi_{j_l}$  be such that  $\Phi_{i_m}(s) = \sup\{\Phi_{i_m}(x) | x \in O(X)\}$ ,  $m = \overline{1, k}$  and  $\Phi_{j_n}(s) = \inf\{\Phi_{j_n}(x) | x \in O(X)\}$ ,  $n = \overline{1, l}$ , where  $\{i_1, \dots, i_k, j_1, \dots, j_l\} \subset \{1, \dots, n_0\}$ ,  $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_l\} = \emptyset$ . It is clear that we can choose distinct points  $s_1, s_2 \in B_\delta(s) \setminus \{s\}$  and continuous functions  $\tilde{\Phi}_{i_1}, \dots, \tilde{\Phi}_{i_k}, \tilde{\Phi}_{j_1}, \dots, \tilde{\Phi}_{j_l}$  such that the following conditions hold: 1)  $\tilde{\Phi}_i(x) = \Phi_i(x)$ ,  $x \in O(X) \setminus B_\delta(s)$ ,  $i \in \{i_1, \dots, i_k, j_1, \dots, j_l\}$ ; 2)  $\tilde{\Phi}_i(s_1) = \sup\{\tilde{\Phi}_i(x) | x \in O(X)\}$ ,  $i \in \{i_1, \dots, i_k\}$  and  $\tilde{\Phi}_i(s_2) = \inf\{\tilde{\Phi}_i(x) | x \in O(X)\}$ ,  $i \in \{j_1, \dots, j_l\}$ ; 3)  $d(\Phi_i, \tilde{\Phi}_i) \leq \varepsilon_1$ , where  $i \in \{i_1, \dots, i_k, j_1, \dots, j_l\}$ . We define the new sets:  $F_1 = F_0 \cup \{\tilde{\Phi}_{i_1}, \dots, \tilde{\Phi}_{i_k}, \tilde{\Phi}_{j_1}, \dots, \tilde{\Phi}_{j_l}\} \setminus \{\Phi_{i_1}, \dots, \Phi_{i_k}, \Phi_{j_1}, \dots, \Phi_{j_l}\}$ ,  $S_1 = S_0 \cup \{s_1\} \setminus \{s\}$ ,  $I_1 = I_0 \cup \{s_2\} \setminus \{s\}$ . We can apply the same procedure to the rest of the points of  $I_0 \cap S_0$  to obtain sets  $I$  and  $S$  of some points of minimums and maximums of another set of functions  $F = \{\tilde{\Phi}_1, \dots, \tilde{\Phi}_{n_0}\}$  with  $I \cap S = \emptyset$  and  $d(\Phi_i, \tilde{\Phi}_i) \leq \varepsilon_1$  for any  $i \in \{1, \dots, n_0\}$ . Note that  $I$ , for example, not necessarily contains *all* points of minimum of every function  $\tilde{\Phi}_i$ , it contains only one such point; the same is about  $S$ . Also we can assume that  $\inf O(X) \notin S$ ,  $\sup O(X) \notin I$  (otherwise we would apply the reasoning from above to these points as well).

Choose function  $\Phi_0 : O(X) \rightarrow \mathbb{R}$  such that  $\Phi_0(I \cup \inf O(X)) \subset \{1\}$  and  $\Phi_0(S \cup \sup O(X)) \subset \{0\}$ . Denote  $Y = \{\pi_\varphi | \varphi \in C(X)\} \cup \{\tilde{\Phi}_i + c_{O(X)} | c \in \mathbb{R}, i = \overline{1, n_0}\}$ . Take any functional  $M \in O^2(X)$ . Let  $M_0 = \{\Lambda \in O^2(X) | \Lambda|_Y = M|_Y, \Lambda(\Phi_0) = 0\}$ , and  $M_1 = \{\Lambda \in O^2(X) | \Lambda|_Y = M|_Y, \Lambda(\Phi_0) = 1\}$ . Due to the choice of function  $\Phi_0$ , we have that  $M_0 \neq \emptyset$ ,  $M_1 \neq \emptyset$ .

Let us show that the mappings  $G_0, G_1 : O^2(X) \rightarrow \exp O^2(X)$  defined by  $G_0(M) = M_0$ ,  $G_1(M) =$

$M_1$  are continuous. Indeed, take any sequence  $\{M_n\}_{n \in \mathbb{N}} \subset O^2(X)$  that converges to some  $M \in O^2(X)$ . We may assume that there exists  $A = \lim_{n \rightarrow \infty} (M_n)_0$ . We must show that the equality  $M_0 = A$  holds. The inclusion  $A \subset M_0$  is obvious. Let us show the inclusion  $M_0 \subset A$  takes place. Assuming the opposite, we get that there are  $\Lambda \in M_0$  and some function  $\Phi \in C(O(X))$  such that  $\Lambda(\Phi) = a > \sup A(\Phi)$  or  $\Lambda(\Phi) = a < \inf A(\Phi)$  (this follows from the fact that all  $(M_n)_0$  are  $O$ -convex, i.e. for any  $V \in O^2(X)$  with  $\inf(M_n)_0 \leq V \leq \sup(M_n)_0$  we have  $V \in (M_n)_0$ , hence their limit is so (see [8])). Suppose the first case holds, for instance. Note that, since  $\mu_O X$  is open, the sequence  $\{\mu_O X^{-1}(\mu_O X(M_n))\}$  converges to  $\mu_O X^{-1}(\mu_O X(M))$ . Hence,  $\sup\{M_n(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}$  and  $\inf\{M_n(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\}$  must converge to  $\sup\{M(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}$  and  $\inf\{M(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\}$  respectively. Indeed, consider any convergent subsequence  $\{\sup\{M_{n_k}(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}\}_{k \in \mathbb{N}}$  of  $\{\sup\{M_n(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}\}_{n \in \mathbb{N}}$ , for example (at least one such subsequence must exist!). Suppose that its limit  $s_1$  is not equal to  $s = \sup\{M(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}$ , say  $s_1 > s$ . Now note that the set  $\mu_O X^{-1}(\nu)$  for any  $\nu \in O(X)$  consists of all possible extensions of the functional  $\bar{\Theta} : D \rightarrow \mathbb{R}$ , where  $D = \{\pi_\varphi \mid \varphi \in C(X)\}$  and  $\bar{\Theta}(\pi_\varphi) = \nu(\varphi)$ . Since any such extension must be order-preserving, its possible values on  $\Phi$  are in the closed interval  $[\sup\{\bar{\Theta}(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}, \inf\{\bar{\Theta}(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\}]$ . So, in our case we get that the possible value of any functional from  $\lim_{k \rightarrow \infty} \mu_O X^{-1}(\mu_O X(M_{n_k}))$  (again we may assume the sequence converges) cannot be less than  $s_1$  on  $\Phi$ , whereas functionals from  $\mu_O X^{-1}(\mu_O X(M))$  are allowed to take any value up to  $s$  on  $\Phi$ , hence  $\{\mu_O X^{-1}(\mu_O X(M_{n_k}))\}_{k \in \mathbb{N}}$  doesn't converge to  $\mu_O X^{-1}(\mu_O X(M))$ , a contradiction with the openness of  $\mu_O X$ . The same reasonings could be applied in the case with the sequence  $\{\inf\{M_n(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\}\}_{n \in \mathbb{N}}$ .

Take now any  $\delta > 0$ . There exists  $k_0 \in \mathbb{N}$  such that  $|\sup\{M_n(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\} - \sup\{M(\pi_\varphi) \mid \pi_\varphi \leq \Phi, \varphi \in C(X)\}| < \delta$ ,  $|\inf\{M_n(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\} - \inf\{M(\pi_\varphi) \mid \pi_\varphi \geq \Phi, \varphi \in C(X)\}| < \delta$  and  $|M(\Phi_i) - M_n(\Phi_i)| < \delta$ ,  $i = \overline{1, n_0}$  for all  $n \geq k_0$ . Hence, we get that  $|\sup\{M_n(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \leq \Phi\} - \sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \leq \Phi\}| < \delta$  and  $|\inf\{M_n(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \geq \Phi\} - \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \geq \Phi\}| < \delta$  for sufficiently large numbers  $n$ . This means that whatever is  $a = M(\Phi) \in [\sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \leq \Phi\}, \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \geq \Phi\}]$ , we can choose  $k_0 \in \mathbb{N}$  such that  $[\sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \leq \Phi\}, \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\}, \Psi \geq \Phi\}] \cap (a - \frac{a - \sup A(\Phi)}{2}, a + \frac{a - \sup A(\Phi)}{2}) \neq \emptyset$  for all  $n \geq k_0$ , which means that we can obtain functionals from  $(M_n)_-$  with values at  $\Phi$  strictly larger than  $\sup A(\Phi)$ , a contradiction. Hence, the mappings  $G_0, G_1$  are continuous.

Take now any  $M \in O^2(X)$  and  $\Lambda \in M_0$ , for instance. We have that  $d_O(M, \Lambda) = \sum_{i=1}^{\infty} \frac{|M(\Phi_i) - \Lambda(\Phi_i)|}{\|\Phi_i\| \cdot 2^i} < \sum_{i=1}^{n_0} \frac{|M(\Phi_i) - \Lambda(\Phi_i)|}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} \leq \sum_{i=1}^{n_0} \frac{|M(\tilde{\Phi}_i) - \Lambda(\tilde{\Phi}_i)| + 4\varepsilon_1}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} = \sum_{i=1}^{n_0} \frac{4\varepsilon_1}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} = \varepsilon$ .

Define now  $g_i : O^2(X) \rightarrow O^2(X)$ ,  $i = \overline{0, 1}$  by the formula  $g_i(M) = \sup G_i$ , for example. Functions  $g_0, g_1$  defined that way are continuous,  $\varepsilon$ -close to  $id_{O^2(X)}$ , with disjoint images and preserve the fibers of  $\mu_O X$ . The theorem is proved.

Now we'll consider the case of  $\tau > \omega$ .

**Theorem 2.2.** *Let  $w(X) = \tau > \omega$ . The map  $\mu_O X : O^2(X) \rightarrow O(X)$  is an  $I^\tau$ -fibration if and only if  $X$  is openly generated and  $\chi$ -homogeneous.*

*Proof. Sufficiency.* Suppose that  $X$  is openly generated and  $\chi$ -homogeneous,  $w(X) = \tau > \omega$ . We'll use theorem 1.6 in combination with lemma 1.1 to prove this part of the statement. Suppose that  $\omega \leq \lambda < \tau$ . Represent  $X$  as the limit of a  $\lambda$ -system  $\mathcal{S} = \{X_\alpha, p_\alpha, \mathcal{A}\}$ , where  $\mathcal{A}$  has the minimal element 1 and  $X_1$  is a singleton. Also we can suppose that all  $p_\alpha$  are open. Consider  $Y_\alpha = O^2(X_\alpha) \times_{O(X_\alpha)} O(X)$ , and by  $q_\alpha$  denote the diagonal product  $q_\alpha = (O^2(p_\alpha), \mu_O X)$ . We obtained a  $\lambda$ -system  $\{Y_\alpha, q_\alpha, \mathcal{A}\}$  with the first limit projection  $q_1$  homeomorphic to  $\mu_O X$ . Note also that every  $q_\alpha$  can be assumed soft since so is  $\mu_O X$ . We will prove that each  $q_\alpha$  allows two disjoint sections.

First let's show that the fibers of each  $q_\alpha$  are infinite. Indeed, consider any  $(\Lambda, \nu) \in Y_\alpha$ . Then  $\mu_O X_\alpha(\Lambda) = O(p_\alpha)(\nu)$ . Denote  $D = \{\pi_\psi \mid \psi \in C(X)\} \cup \{\Phi \circ O(p_\alpha) \mid \Phi \in C(O(X_\alpha))\}$ . All mappings  $q_\alpha$ , being soft, are surjective. Hence, there's at least one functional  $\Theta : C(O(X)) \rightarrow \mathbb{R}$  such that  $q_\alpha(\Theta) = (\Lambda, \nu)$ . Then  $\Theta(\pi_\psi) = \nu(\psi)$ ,  $\Theta(\Phi \circ O(p_\alpha)) = \Lambda(\Phi)$ .

Our present aim is to find a function  $\Phi_0 \in C(O(X))$  such that there would exist at least two distinct extensions of  $\Theta|_D$  on the space  $D \cup \{\Phi_0 + c_{O(X)} \mid c \in \mathbb{R}\}$ .

Since  $X$  is  $\chi$ -homogeneous and  $w(X_\alpha) < w(X)$ , the mapping  $p_\alpha$  doesn't have one-point fibers, and so doesn't  $O(p_\alpha)$  (theorem 1.3). Denote  $S = \{\sup O(p_\alpha)^{-1}(\lambda) \mid \lambda \in O(X_\alpha)\}$ ,  $I = \{\inf O(p_\alpha)^{-1}(\lambda) \mid \lambda \in O(X_\alpha)\}$ . Both these sets are closed due to the openness of  $O(p_\alpha)$  and operations  $\sup, \inf : \exp O(X) \rightarrow O(X)$  being continuous. Now define  $\Phi_0 \in C(O(X))$  to be a function with  $\Phi_0(S) = 0$  and  $\Phi_0(I) = 1$ . Suppose that  $\Phi \circ O(p_\alpha) \leq \Phi_0$ . Since  $\Phi \circ O(p_\alpha)$  is constant on the fibers of  $O(p_\alpha)$ , this implies  $\Phi \circ O(p_\alpha) \leq 0$ , hence  $\Phi \leq 0$  and  $\Theta(\Phi \circ O(p_\alpha)) = \Lambda(\Phi) \leq 0$ . Similarly,  $\Theta(\Phi \circ O(p_\alpha)) \geq 1$  for any  $\Phi \circ O(p_\alpha) \geq \Phi_0$ . Now pick any  $\psi \in C(X)$  with  $\pi_\psi \leq \Phi_0$ , for example. We have that  $\nu \in O(p_\alpha)^{-1}(\lambda)$  for some  $\lambda \in O(X_\alpha)$ . Then  $\Theta(\pi_\psi) = \pi_\psi(\nu) \leq \pi_\psi(\sup(O(p_\alpha)^{-1}(\lambda))) \leq \Phi_0(\sup(O(p_\alpha)^{-1}(\lambda))) = 0$ . Similarly,  $\Theta(\pi_\psi) \geq 1$  for all  $\pi_\psi \geq \Phi_0$ . Also, it is obvious that  $\Phi_0$  doesn't belong to  $D$ , hence, if we define  $\Theta(\Phi_0 + c_{O(X)}) = a + c$ , where  $a \in [0, 1]$ , we'll obtain an order-preserving functional on  $D \cup \{\Phi_0 + c_{O(X)} \mid c \in \mathbb{R}\}$ , which we can extend on the whole space  $C(O(X))$  according to lemma 2 of [7]. Therefore, we've shown that  $\Theta$  has at least two extensions from  $D$ , hence the fibers of  $q_\alpha$  are not singletons. So, for any  $(\Lambda, \nu) \in Y_\alpha$  define  $g_1 = \inf q_\alpha^{-1}(\Lambda, \nu)$ ,  $g_2 = \sup q_\alpha^{-1}(\Lambda, \nu)$ . The mappings  $g_1, g_2$  are continuous disjoint sections for  $q_\alpha$ .

Hence, the mapping  $\mu_O X$  satisfies the condition of lemma 1.1, and by theorem 1.6 it is an  $I^\tau$ -fibration.

*Necessity.* Since  $\mu_O X$  is an  $I^\tau$ -fibration, we have that it is soft. The softness of  $\mu_O X$  implies that  $X$  is openly generated (theorem 1.2) The space  $X$  must be  $\chi$ -homogeneous, since, if we suppose the opposite, we get that  $O^2(X)$  is not  $\chi$ -homogeneous (theorem 1.4), hence, there exist some  $\Lambda \in O^2(X)$  with  $\chi(\Lambda, O^2(X)) = \tau' < \tau$ , and therefore  $\mu_O X^{-1}(\mu_O X(\Lambda))$  is not homeomorphic to  $I^\tau$ . The theorem is proved.

**Note.** Proofs of theorems 2.1 and 2.2 are the same in case of monad  $\mathbb{O}H$ . Note that proof of theorem 2.1 (the part of it which concerns the choice of function  $\Phi_0$ ) could be a bit easier for monad  $\mathbb{O}$ . Indeed, the function  $\Phi_0 : O(X) \rightarrow \mathbb{R}$  such that  $\Phi_0(\inf O(X)) = 2\alpha$  and  $\Phi_0(\sup O(X)) = -2\alpha$ , where  $\alpha = \max\{\sup \Phi_i - \inf \Phi_i \mid i = \overline{1, n_0}\}$  would do. In this case we can take  $M_0 = \{\Lambda \in O^2(X) \mid \Lambda|_Y = M|_Y, \Lambda(\Phi_0) = -\alpha\}$ , and  $M_1 = \{\Lambda \in O^2(X) \mid \Lambda|_Y = M|_Y, \Lambda(\Phi_0) = \alpha\}$  for any  $M \in O^2(X)$ , where

$Y = \{\pi_\varphi \mid \varphi \in C(X)\} \cup \{\tilde{\Phi}_i + c_{O(X)} \mid c \in \mathbb{R}, i = \overline{1, n_0}\}$ . But in the case of  $OH$  the argumentation in proof of theorem 2.1 with  $\Phi_0$ ,  $M_0$  and  $M_1$  as just described fails.

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